Probability Model of a Taxi Stand

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Outline

- Queueing models
- Our Problem
- The Solution
- In Conclusion
- Future Work
- Acknowledgments

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Queueing models

What is a Queueing Model?
A probabilistic description of a queue at a service location.

The standard queuing model assumes that customers arrive at a random rate and line up for service. There is a single server and the length of service is random.

Thus, a Queuing Model consists of:

(i) The rate of customer arrival
(ii) The length of service
(iii) The number of servers (in more general cases)

What is the purpose of a Queueing Model?

(i) Predict the queue's probable and expected length
(ii) Find the probable and mean waiting time
(iii) Estimate the server's idle periods

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The M/M/1 Model

A common queueing model is known as the M/M/1 queue.

- M- Customers arrive according to a Poisson process (which is a Markov process)
- M- The length of service is a random variable with an exponential distribution (also Markov)
- 1- A single server

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Our Problem - The Taxi Station

We decided to generalize the standard queue model to a situation where both the customers and the servers have to line up, making it a dual queueing system.

A good application would be a taxi station:

- People who need a ride arrive at a taxi station and line up for the next available taxi
- Taxis arriving at the station line up for customer pick up
- Customers and taxis are matched on a first come first serve basis

Additionally,

- Customers who find too many people ahead of them, turn around and leave.
- Likewise, taxi drivers who find too many taxis ahead of them, leave

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Our Goals

We want to answer the following questions:

- What is the proportion of time either line is empty?
- What is the fraction of customers who will catch a taxi?
- What is the fraction of taxis that will pick up a customer?
- What is the expected wait time in each of the lines?

We need to extend the typical model in order to answer these questions because there is more than one queue.

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Model Parameters

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We have the following parameters:

$$\lambda,\ N,\ \mu,\ M$$

- People arrive as a Poisson process with rate λ (per hour)
- People balk when they find N customers ahead of them
- Taxis arrive as a Poisson process with rate μ (per hour)
- Taxis balk when they find M taxis ahead of them

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The Analysis

We let $P_{i,j}$ denote the proportion of time the system is in state (i, j), where i is the number of customers and j is the number of taxis waiting.

If a customer arrives and there is a taxi waiting, they depart immediately.

Therefore, $P_{i,j} = 0$ whenever $i > 0$ and $j > 0$

Thus,

$$\sum_{i=0}^{N}\sum_{j=0}^{M} P_{i,j} = \sum_{i=0}^{N} P_{i,0} + \sum_{j=1}^{M} P_{0,j} = 1$$

Our first step will be to obtain an expression for $P_{0,0}$.

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Balance Equations: Customers

Borrowing from fluid dynamics, the rate of entering a given state must equal the rate of leaving that state.

$$
\begin{aligned}
\underline{\textit{In}} \qquad & \qquad \underline{\textit{Out}} \\
\lambda P_{0,1} + \mu P_{1,0} &= (\lambda+\mu)P_{0,0} \\
\lambda P_{0,0} + \mu P_{2,0} &= (\lambda+\mu)P_{1,0} \\
&\vdots \\
\lambda P_{i-1,0} + \mu P_{i+1,0} &= (\lambda+\mu)P_{i,0} \\
&\vdots \\
\lambda P_{N-2,0} + \mu P_{N,0} &= (\lambda+\mu)P_{N-1,0} \\
\lambda P_{N-1,0} &= \mu P_{N,0}
\end{aligned}
$$

It can be shown that

$$P_{i,0} = \frac{\lambda}{\mu} P_{i-1,0} \text{ for } 0 < i \leq N$$

Balance Equations: Taxis

Similarly, we have:

$$\begin{aligned}
\underline{\textit{In}} \quad & \quad \underline{\textit{Out}} \\
\lambda P_{0,1} + \mu P_{1,0} &= (\lambda + \mu) P_{0,0} \\
\mu P_{0,0} + \lambda P_{0,2} &= (\lambda + \mu) P_{0,1} \\
&\vdots \\
\mu P_{0,j-1} + \lambda P_{0,j+1} &= (\lambda + \mu) P_{0,j} \\
&\vdots \\
\mu P_{0,M-2} + \lambda P_{0,M} &= (\lambda + \mu) P_{0,M-1} \\
\mu P_{0,M-1} &= \lambda P_{0,M}
\end{aligned}$$

Likewise,

$$P_{0,j} = \tfrac{\mu}{\lambda} P_{0,j-1} \text{ for } 0 < j \leq M.$$

The Solution

Which yields the following:

$$
\begin{aligned}
1 &= \sum_{i=0}^{N} P_{i,0} + \sum_{j=1}^{M} P_{0,j} \\
&= P_{0,0} \sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^{i} + P_{0,0} \sum_{j=1}^{M-1} \left(\frac{\mu}{\lambda}\right)^{j} \\
&= P_{0,0} \left(\frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\frac{\lambda}{\mu}} + \frac{1-\left(\frac{\mu}{\lambda}\right)^{M}}{1-\frac{\mu}{\lambda}} \right) \\
&= P_{0,0} \left(\frac{\lambda\left(\frac{\lambda}{\mu}\right)^{N} - \mu\left(\frac{\mu}{\lambda}\right)^{M}}{\lambda-\mu} \right)
\end{aligned}
$$

Therefore,

$$
P_{0,0} = \frac{\lambda-\mu}{\lambda\left(\frac{\lambda}{\mu}\right)^{N} - \mu\left(\frac{\mu}{\lambda}\right)^{M}}
$$

What proportion of time is either line empty?

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The customer queue is empty whenever there is either a taxi in queue or no one is waiting at the station.

So,

$$P(\text{No customer}) = \sum_{j=0}^{M} P_{0,j} = P_{0,0} \sum_{j=0}^{M} (\tfrac{\mu}{\lambda})^j = \frac{\lambda - \mu\left(\frac{\mu}{\lambda}\right)^M}{\lambda\left(\frac{\lambda}{\mu}\right)^N - \mu\left(\frac{\mu}{\lambda}\right)^M}$$

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The Proportion of Customers Served

We are concerned with what proportion of customers will catch a taxi vs the proportion who will balk.

The proportion of customers who will catch a taxi equals

$$1 - P(\text{Customer queue is at capacity})$$

We denote λ_{queue} = rate of customers catching taxis

$$\lambda_{queue} = \lambda(1 - P_{N,0}) = \lambda(1 - P_{0,0}(\frac{\lambda}{\mu})^N)$$

Similarly,

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The Expected Wait Times

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Little's Theorem:

$$E[\text{Wait Time}] = \frac{E[\text{Line Length}]}{\lambda_{queue}}$$

Here, we use Little's Theorem.

Let W and V denote the wait times for customers and taxis respectively. Thus, we have:

$$E[W] = \frac{E[\#\text{ of customers}]}{\lambda_{queue}} = \frac{\sum_{i=0}^{N} i \cdot P_{i,0}}{\lambda(1 - P_{N,0})} = \frac{P_{0,0} \sum_{i=0}^{N} i(\frac{\lambda}{\mu})^i}{\lambda(1 - P_{0,0}(\frac{\lambda}{\mu})^N)}$$

$$E[V] = \frac{E[\#\text{ of taxis}]}{\mu_{queue}} = \frac{\sum_{j=0}^{M} j \cdot P_{0,j}}{\mu(1 - P_{0,M})} = \frac{P_{0,0} \sum_{j=0}^{M} j(\frac{\mu}{\lambda})^j}{\mu(1 - P_{0,0}(\frac{\mu}{\lambda})^M)}$$

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Examples

Here are some numerical examples:

When taxis arrive faster than customers:

Customers	Taxis
The random arrival rate of Customers, λ, is {4} per hour.	The random arrival rate for taxis, μ, is {5} per hour.
The maximum for the line of Customers, m, is {2}.	The maximum for the line of taxis, n, is {3}.
The proportion of time there are no customers is {0.66138}.	The proportion of time there are no taxis is {0.51210}.
The expected line length is {0.62729} Customers.	The expected line length is {0.75896} taxis.
The expected wait time is {10.327} minutes.	The expected wait time is {12.494} minutes.
The fraction of customers that take taxis is {0.91118}.	The fraction of taxis that take Customers is {0.72894}.
The proportion of time spent in the empty state P0 is {0.17348}.	To Check that these probabilities add to 1: {1.0000}

When the customers arrive faster than taxis:

Customers	Taxis
The random arrival rate of Customers, λ, is {5} per hour.	The random arrival rate for taxis, μ, is {4} per hour.
The maximum for the line of Customers, m, is {3}.	The maximum for the line of taxis, n, is {2}.
The proportion of time there are no customers is {0.51210}.	The proportion of time there are no taxis is {0.66138}.
The expected line length is {0.75896} Customers.	The expected line length is {0.62729} taxis.
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The proportion of time spent in the empty state P0 is {0.17348}.	To Check that these probabilities add to 1: {1.0000}

When the rates are equal:

Customers	Taxis
The random arrival rate of Customers, λ, is {10} per hour.	The random arrival rate for taxis, μ, is {10} per hour.
The maximum for the line of Customers, m, is {2}.	The maximum for the line of taxis, n, is {2}.
The proportion of time there are no customers is {0.60000}.	The proportion of time there are no taxis is {0.60000}.
The expected line length is {0.60000} Customers.	The expected line length is {0.60000} taxis.
The expected wait time is {4.5000} minutes.	The expected wait time is {4.5000} minutes.
The fraction of customers that take taxis is {0.80000}.	The fraction of taxis that take Customers is {0.80000}.
The proportion of time spent in the empty state P0 is {0.20000}.	To Check that these probabilities add to 1: {1.0000}

When the rates are almost equal:

Customers	Taxis
The random arrival rate of Customers, λ, is {9} per hour.	The random arrival rate for taxis, μ, is {9.3} per hour.
The maximum for the line of Customers, m, is {2}.	The maximum for the line of taxis, n, is {2}.
The proportion of time there are no customers is {0.619556}.	The proportion of time there are no taxis is {0.580229}.
The expected line length is {0.567548} Customers.	The expected line length is {0.633097} taxis.
The expected wait time is {4.65453} minutes.	The expected wait time is {5.19211} minutes.
The fraction of customers that take taxis is {0.812896}.	The fraction of taxis that take Customers is {0.786674}.
The proportion of time spent in the empty state P0 is {0.199785}.	To Check that these probabilities add to 1: {1.}

In Conclusion

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Once we had achieved a closed form of this model we could:

- Accurately predict the length of either queue
- Write a program to perform heavy calculations
- Answer general questions about a dual capacity stochastic queue!

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Future Work

With this model, a natural extension would be to:

- Incorporate customer and taxi reneging, where either a customer or taxi in the queue may leave if they wait too long
- Allow the lines to have an infinite capacity

And there are many other generalizations.

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Project Advising

We would like to thank Dr. Shahar Boneh for taking the role of our project advisor.

Reference

Ross, S. (2014). Introduction to Probability Models

Thank You!

Questions?

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