An Exploration of Iterative Matrix Transformations

Pikes Peak Regional Undergraduate Mathematics Conference

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Topics

For an Independent study, we decided to take a much deeper look into Linear Algebra. We have:

- a Mathematica program to visualize and compute measures for numerical evaluation for the Power Method!
- created a scheme to generalize the Power Method to rectangular matrices!
- established and proved a conjecture for the characteristic polynomials for symmetric matrices!

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \ldots + C_1\lambda + C_0$$

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What is it?

The Power Method is:

- A numeric method for finding eigenvalues
- Very easy to program
- Has many methods spawning from it (i.e. Inverse PM)

While playing around with this method, we discovered an interesting property,

- Geometrically, objects were being rotated toward the eigenspace!

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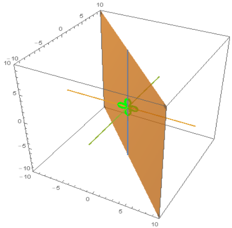

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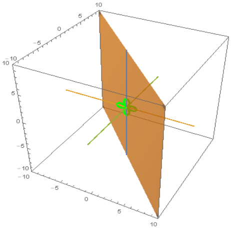

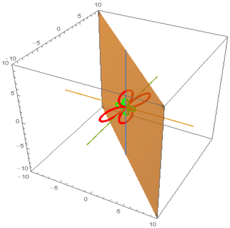

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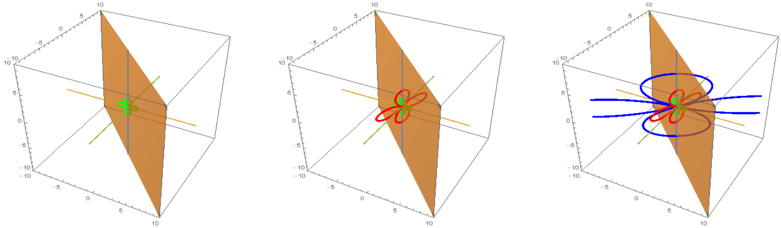

Using the Power Method

For the Power Method to work, we must have that:

- The matrix A must be square.
- The matrix A has a distinct absolute greatest eigenvalue.

Method:

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 = A^2\mathbf{x}_0 \\ &\vdots \\ \mathbf{x}_n &= A\mathbf{x}_{n-1} = A^n\mathbf{x}_0 \end{aligned}$$

Rayleigh's Quotient:

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

Convergence: Given that the eigenvalues are ordered and $|\lambda_2| < |\lambda_1|$, the convergence rate S is:

$$S = \frac{|\lambda_2|}{|\lambda_1|}$$

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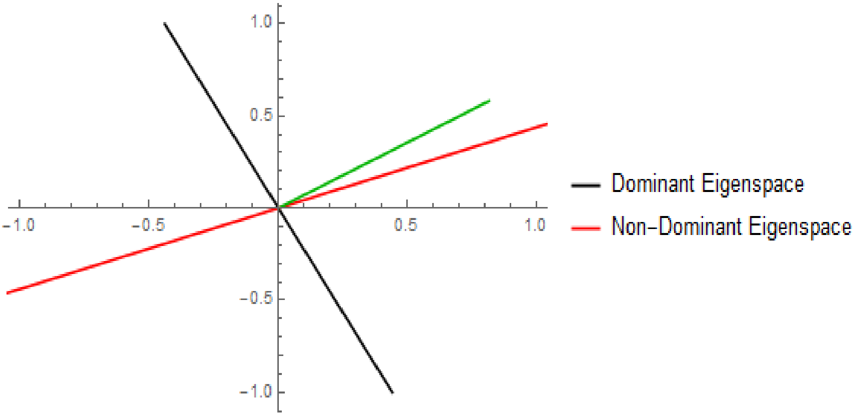

Rectangular Matrices

More commonly seen, rectangular matrices differ from square in many ways

- The number of columns and rows are not equal
- They can change the dimension of the object's ambient space
- They don't have eigenvalues

Instead, they have singular values, σ. Where σ is found to be the square root of the eigenvalues of $AA^{\top}$.

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$$\underset{m \times n}{\overset{A}{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}}} = \underset{m \times r}{\overset{U}{\begin{bmatrix} u_{1,1} & \cdots & u_{1,r} \\ \vdots & \ddots & \vdots \\ u_{m,1} & \cdots & u_{m,r} \end{bmatrix}}} \cdot \underset{r \times r}{\overset{\Sigma}{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}}} \cdot \underset{r \times n}{\overset{V^{\mathrm{T}}}{\begin{bmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,r} & \cdots & \sigma_{n,r} \end{bmatrix}}}$$

Generalizing the Power Method

Products of Transposes:

- $A^{\mathsf{T}}A$ vs. AA^{T}
- Used in Singular Value Decomposition

Connections to the Power Method:

- We could use the PM on $A^{\mathsf{T}}A$ to find the singular values
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Applications

- We can choose the smaller of $A^{\mathsf{T}}A$ or AA^{T} to perform calculations!

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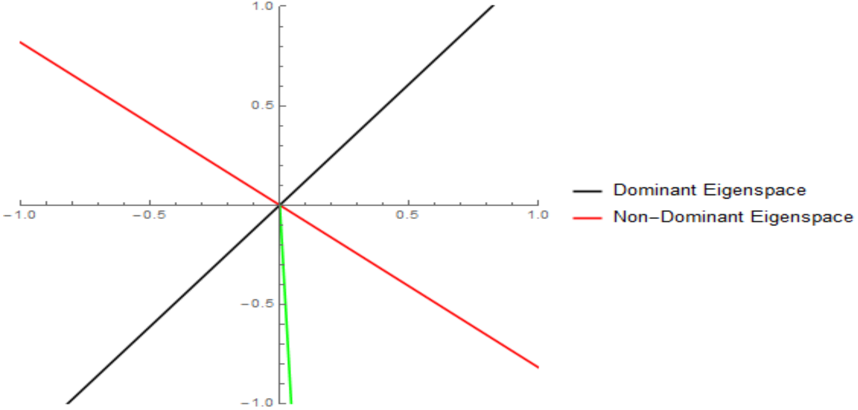

Our Theorem

Theorem (G, P. 2015)

Let A be an $m \times n$ matrix with $m < n$ and rank(A)=k, then A^TA and AA^T are symmetric matrices and their characteristic polynomials are

$$(-1)^n\lambda^{n-k}\left(\lambda^k + C_{n-1}\lambda^{k-1} + ... + C_{n-(k-1)}\lambda + C_{n-k}\right)$$

and

$$(-1)^m\lambda^{m-k}\left(\lambda^k + C_{n-1}\lambda^{k-1} + ... + C_{n-(k-1)}\lambda + C_{n-k}\right)$$

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Computing the Coefficients

For an $n \times n$ matrix A and $1 \leq k \leq n$:

Paul Horst Method

Each coefficient C_{n-k} is equal to the sum of the k^{th} order principal minors of A.

Faddeev-Leverrier Algorithm

Construct a sequence of matrices $\{\mathrm{M}_k\}$ and calculate each coefficient C_{n-k} by computing the trace of $[AM_k]$.

$$M_0 = 0 \qquad C_n = 1$$

$$M_k = AM_{k-1} + C_{n-k+1}\mathrm{I}_n \qquad C_{n-k} = -\frac{1}{k}\mathrm{Tr}(AM_k)$$

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Establishing Lemmas

Lemma 1:

If A, B are $n \times n$ matrices and c is a scalar, then

$$\text{Tr}(cA + B) = c\text{Tr}(A) + \text{Tr}(B)$$

Lemma 2:

If C is an $m \times n$ matrix and D is an $n \times m$ matrix,

$$\text{Tr}(CD) = \text{Tr}(DC)$$

Lemma 3:

If C is an $m \times n$ matrix and D is an $n \times m$ matrix,

$$\text{Tr}\left[(CD)^k\right] = \text{Tr}\left[(DC)^k\right]$$

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Overview of Proof

The Characteristic Polynomials

$$A^{\mathsf{T}}A : \lambda^n + C_{n-1}\lambda^{n-1} + ... + C_1\lambda + C_0$$
$$AA^{\mathsf{T}} : \lambda^m + D_{m-1}\lambda^{m-1} + ... + D_1\lambda + D_0$$

- With Paul Horst's method and the definitions of rank and determinental rank, we conclude these polynomials have the same number of terms.

  $$A^{\mathsf{T}}A : (-1)^n\lambda^{n-k}\left(\lambda^k + C_{n-1}\lambda^{k-1} + ... + C_{n-(k-1)}\lambda + C_{n-k}\right)$$
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- Using the Fadeev-Leverrier algorithm and the lemmas, we can establish that each remaining $C_{n-i} = D_{m-i}$, thus proving that the two polynomials are identical sans a factor of λ^{n-m}.

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Future Work

- What must be considered in the case where the matrix has no distinct dominant eigenvalue
  - Similarly, for a rectangular matrix with no distinct dominant singular value
- Explore the combinatorial relation between the entries in the product of transposes
- Look further into the Faddeev-Leverrier Algorithm

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Credits

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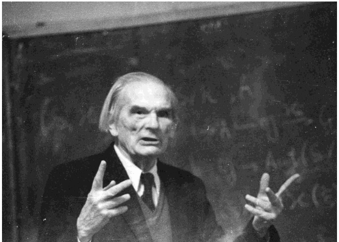
- Our math department for incredible support
- Our advisor Dr. Diane Davis

We wouldn't be here today without the the backing we received over this study!

Leverrier

Faddeeva

Faddeev



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