Numerical Solutions to Stochastic Differential Equations

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Outline

- Stochastic Differential Equations
- Methods of Solution
- Example
- Orders of Approximation
- Future Work

Stochastic Process

What is it?

Coming from Probability Theory, a stochastic process is a collection of random variables that represent the evolution of a system over time.

You may of heard of:

- Markov Chains
- Queueing Theory
- Renewal Theory
- Brownian Motion

These are all stochastic processes!

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Stochastic Differential Equations

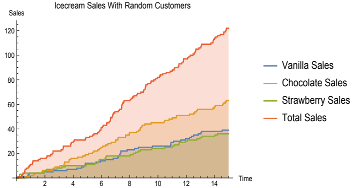
As we know, differential equations are useful tools in modeling real world behaviors like:

- An oscillating spring
- the trajectory of a object

Incorporate this idea with the random aspect of stochastic process, and we get a real world behavior that may have too many factors to predict in a deterministic fashion!

- Population growths with disease or natural disaster mechanics
- Queue lengths and expected wait times
- Game strategy (Stock market, betting, etc.)

Poisson Icecream Sales



Brownian Motion

A brief history,

- 1827- Initially discovered by a botanist, Robert Brown, when he noticed the movement of pollen in water to be very "random" and seemingly unexplainable
- 1905- Albert Einstein published a paper explaining Brown's observation with a mathematical model which helped solidify the theory of atoms and molecules
- 1908- Jean Perrin verified Einstein's model experimentally

A Brownian Motion is a continuous-time stochastic process which can be looked at as the limiting process of a random walk. The derivative of a brownian motion is often referred to as white noise, which can be introduced into a ODE to gain insight.

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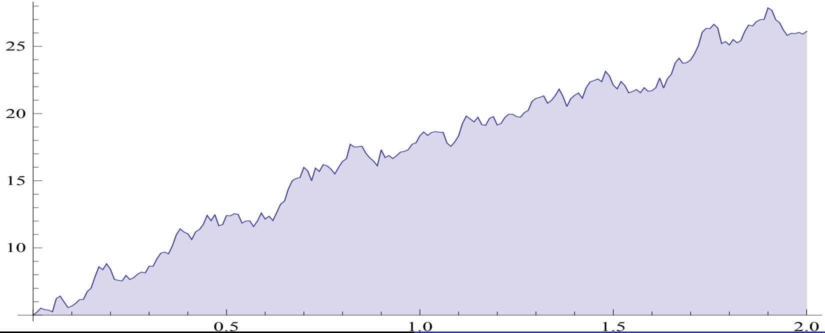
Brownian Motion

A Brownian Motion, B_t, can be approximated with a normal distribution if the time intervals are small enough.

Identity

$$\Delta B_i \approx z_i \sqrt{\Delta t_i}$$



The Euler-Maruyama Method

For first order Stochastic differential equations (SDEs), we can always reduce down to the form*:

$$dy = f(t, y)dt + g(t, y)dB_t$$

The Euler-Maruyama Method (Order $\frac{1}{2}$):

Denote $f_i = f(t_i, w_i)$ and $g_i = g(t_i, w_i)$,

$$w_0 = y_0$$

$$w_{i+1} = w_i + f_i \Delta t_i + g_i \Delta B_i$$

(Recall $\Delta B_i \approx z_i \sqrt{\Delta t_i}$, this is where our order of $\frac{1}{2}$ comes from)

Other Methods

The Milstein Method (Order 1):

By taking Euler's out another term in the Taylor expansion,

$$w_0 = y_0$$

$$w_{i+1} = w_i + f_i \Delta t_i + g_i \Delta B_i + \frac{1}{2} g_i \cdot \frac{\partial g_i}{\partial y}((\Delta B_i)^2 - \Delta t_i)$$

First Order Runge-Kutta (Order 1):

Using a forward difference approximation on Milstein's,

$$w_0 = y_0$$

$$w_{i+1} = w_i + f_i \Delta t_i + g_i \Delta B_i + \frac{1}{2\sqrt{\Delta t_i}}[g(t_i, w_i + g_i\sqrt{\Delta t_i}) - g_i]((\Delta B_i)^2 - \Delta t_i)$$

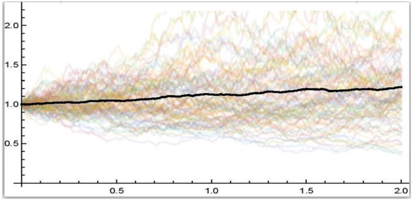
Example

A Geometric Brownian Motion

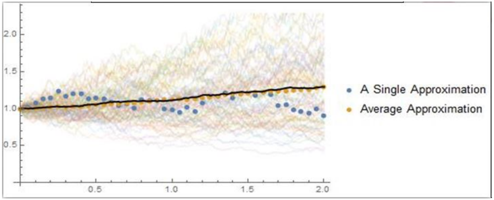
Consider $$dy = \mu y dt + \sigma y dB_t$$

Solution: $$y = y_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Choosing $\mu = .1$, $\sigma = .1$ and $y_0 = 1$, and using Euler's,



100 realizations of true solution.



Approximations with step size .05.

Orders of Approximation

These standard methods of approximating SDEs have relatively low orders of approximation compared to ODEs. For example Eulers method is order 1 for ODEs but $\frac{1}{2}$ for SDEs.

A few reasons:

- No predictable patterns in Stochastic Process
- The error is a random variable
- Each realization will be different

An order of $\frac{1}{2}$ for a SDE approximation behaves quite well. Higher orders are possible at the cost of intense calculations.

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Notes

Itō's Lemma

If $y = f(t,x)$, then

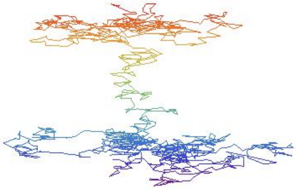
$$dy = \frac{\partial f}{\partial t}(t,x)\,dt + \frac{\partial f}{\partial x}(t,x)\,dx + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)\,dx\,dx$$

Where $dt\,dt = 0$, $dt\,dB_t = dB_t\,dt = 0$, and $dB_t\,dB_t = dt$

*-Using this lemma from Itō Calculus, we can reduce a SDE to

$$dy = f(t,y)dt + g(t,y)dB_t$$

Future Work



- Re-write the Mathematica program!!!!
- Further research into probability measures and stochastic calculus
- Perhaps more work in relating numerical and stochastic process through the use of functional spaces.

Thank you!

Questions?